TechS Vidya e-Journal of Research
ISSN 2322 - 0791
Vol. 1 (2012-13) pp. 88-108.

On Sangamagrāma Mādhava's (c.1350 - c.1425 CE) algorithms for the computation of sine and cosine functions[1]

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Abstract: Sangamagrāma Mādhava (c.1350 - 1425 CE), the founder of the Kerala School of Astronomy and Mathematics, is credited with the discovery of the power series expansions of the trigonometric sine and cosine functions more than two centuries before these series were rediscovered in Europe by Sir Isaac Newton and Gottfried Wilhelm Leibniz in c.1665 CE. This paper examines and analyzes Mādhava's contribution from an algorithmic point of view and shows that modern algorithms for the computation of the values of sine and cosine functions employ essentially the same ideas and methods used by Mādhava for the computation of the values of these functions. For this purpose Mādhava's algorithms have been compared with modern computer programmes for the evaluation of these functions which are included in standard run time C libraries. The paper also observes that Mādhava had used Horner's scheme for the evaluation of polynomials, a fact that has not been noticed by researchers so far. The descriptions of the power series expansions of the sine and cosine functions in Mādhava's own words and also their rendering in modern notations are discussed in this paper. The sine table computed by Mādhava has also been reproduced. The paper concludes by stressing the necessity of making further investigations into the contributions of the Kerala School of Astronomy and Mathematics from an algorithmic point of view. This paper is also a tribute to the computational genius of the Kerala School which flourished in a geographical area very proximal to the venue of this International Conference.

Keywords: Kerala school of astronomy and mathematics, Madhava, sine, cosine, series expansions

[1]A preliminary version of this paper was presented in the International Conference on Computational Engineering, Practices & Techniques, November 2010, held at MES College of Engineering, Kuttippuram. A revised version of the paper was published in MES Journal of technology & Management, Volume 01, No. 02. The authors are thankful to the Editors of the MES Journal of technology & Management for giving them permission to publish the paper in TechS Vidya e-Journal of Research.

1 Introduction

The trigonometric sine and cosine functions are two elementary transcendental functions in mathematics which are extensively used in all areas of science and engineering. Development of methods for the accurate and fast computation of the values of these functions is an area of much interest to numerical analysts and computationalists. In this context it is highly interesting to explore how this problem was tackled in pre-modern times. It would come as a great surprise for many that nearly six centuries ago, even before Sir Isaac Newton was born, mathematicians in remote villages in the Indian state of Kerala had developed computational schemes essentially similar to modern algorithms for the computation of the values of these functions. Sangamagrāma Mādhava (c.1350 - c.1425 CE) was the pioneer of these developments. He discovered the power series expansion of the sine and cosine functions, developed the Horner's scheme for the computation of polynomials nearly four centuries before the scheme was invented in Europe, and successfully applied these results to generate a table of values of the sine function accurate to about eight places of decimals.

This paper discusses these developments. After this introductory Section 1, in Section 2 of the paper we briefly consider the historical context of these ideas. We have introduced the main characters of the Kerala School of Astronomy and Mathematics and also the geographical area where they had flourished. In Section 3 we have explained some of the relevant terminology used by pre-modern mathematicians of India. This includes a discussion of the Kaṭapayādi scheme which is a method for the representation of numbers using letters of the Sanskrit alphabet. Section 4 is an introduction to Mādhava's method for the computation of the value of π. In Section 5, 6 and 7, we present English translations of Sanskrit passages containing descriptions of the power series expansions of sine and cosine functions and their rendering in modern notations. In Section 8 we have reproduced Mādhava's sine table. Section 9 is devoted to an analysis of Mādhava's algorithm from a modern perspective. In Section 10, we compare Mādhava's algorithm for the computation of sine and cosine functions with the current standard schemes for the computation of these functions. We conclude the paper in Section 11 by stressing the necessity for more intensive study of these ideas which is likely to lead to better insights into the understanding of modern computational schemes. In the transliteration of the names and words originally given in Sanskrit, we have followed ISO 15919 standard which is an extension of International Alphabet of Sanskrit Transliteration (IAST). The relevant details of these schemes are given in the Appendix section of this paper.

2 Kerala school of astronomy and mathematics

Even though the contributions of the mathematicians of the "classical period" of Indian mathematics are well known, the contributions of mathematicians of a later period are not so well known (see, for example, [1]). Many historians of mathematics were of the belief that creation of new mathematical ideas in India generally ceased to exist after Bhāskara II. However, research in the past half century has revealed the flourishing of a remarkable galaxy of mathematicians and astronomers in Kerala who had made substantial contributions to the development of mathematics. C.M. Whish, a Civil Servant with the East India Company, had already reported the existence of such a group of personages and their accomplishments (see [2]).[2] But his paper was generally ignored by scholars. These mathematicians and astronomers, who lived over a period from the fourteenth to eighteenth century CE are now collectively referred to as forming the Kerala School of Mathematics. For a detailed discussion of the mathematical and astronomical accomplishments of the Kerala School one may refer to [1], [4] and [5].

2.1 Prominent members of the Kerala School of astronomy and Mathematics

It is generally agreed that Sangamagrāma Mādhava who flourished during c.1350 - c.1425 CE is the founder of this School. No personal details about Mādhava has come to light (see [6]). The appellation Sangamagrāma is surmised to be a reference to his place of residence and some historians has identified it as modern day Irinjālakuda in Thrissur District in Kerala. Only one or two minor works of Mādhava have survived. But there are copious references and tributes to Mādhava in the works of other authors about whom accurate details are available. It is these references that establish conclusively that Mādhava was the founder of the Kerala School.

Paramesvara (c.1380 - c.1460), a pupil of Sangamagrāma Mādhava, was instrumental in introducing the Drigganita system of astronomical computations in Kerala. Dāmodara, another prominent member of the Kerala school, was his son and also his pupil. Paramesvara was also a teacher of Nīlakaṇṭha Somayāji (1444 - 1544).

One of the most influential works of Nīlakantha Somayāji was the comprehensive astronomical treatise Tantrasamgraha completed in 1501 (see [7]). He had also composed an elaborate commentary on Āryabhaṭīya called the Āryabhaṭīya Bhāṣya. In this Bhāṣya, Nīlakaṇṭha had discussed infinite series expansions of trigonometric functions and problems of algebra and spherical geometry. Grahaparīkṣakrama is a manual on making observations in astronomy based on instruments of the time.

Jyēṣṭhadēva (c.1500 - c.1600) was another prominent member of the Kerala School. He is most famous for his work titled Yuktibhāṣa (see [8]). The work was unique in several aspects. Firstly, it was composed in Malayālam a regional language of the Indian state of Kerala. This was in opposition to the then prevailing practice of writing technical works in Sanskrit, the language of scholarship. Secondly, it contained elaborate proofs of the results expounded in the treatise. Incidentally this treatise proves that the idea of proof is not alien to the mathematical traditions of India. Yuktibhāṣa also contains clear statements of the power series expansions of the sine and cosine functions and also lucid expositions of their proofs.

Citrabhānu (c.1550), Śankara Vāriar (c.1500 - c.1560, author of Kriya-kramakari), Acyuta Piṣāraṭi (fl.1550 - 1621), Putumana Sōmayāji who authored Karaṇapadhati and Śankara Varman (1774 - 1839), author of Sadratnamāla, were some of the subsequent promi-

[2]This has been reproduced in [3].

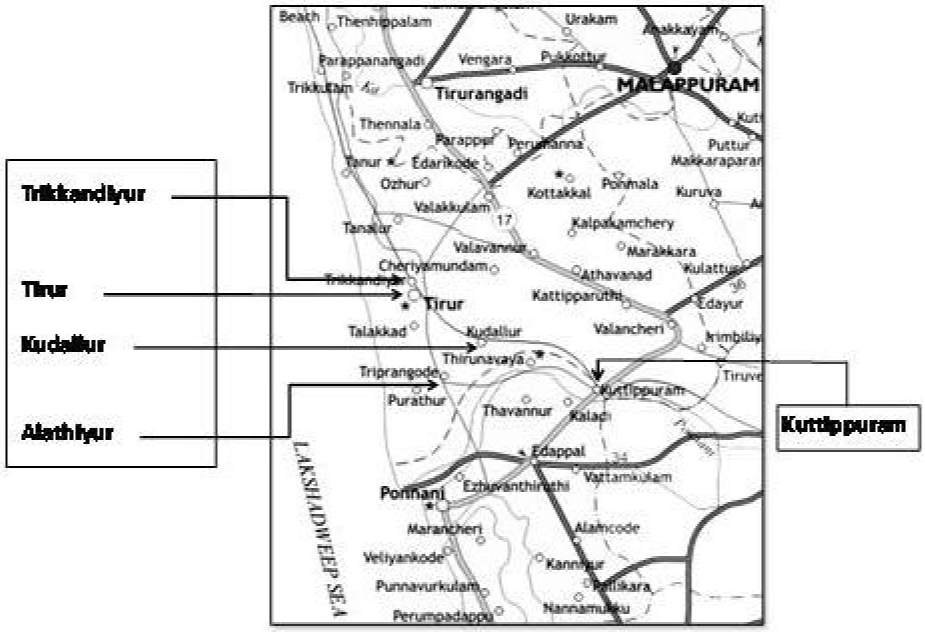

Figure 1: The geographical area where the Kerala School of Mathematics and Astronomy flourished

nent members of the Kerala School of Astronomy and Mathematics.

2.2 Geographical area of the Kerala School

As has already remarked, the place where Madhava, the founder of the Kerala School, flourished is not known exactly. However, it is known with certainty that Parameśvara, Dāmōdara, Nīlakanṭha Sōmayāji, Jyēṣhṭhadēva and Acyuta Pishāraṭi all flourished in an area around modern day Tirur, Trikkandiyur, Alathiyur and Kudallur in the Malppuram District in Kerala. These are not far away from Kuttippuram. (See Figure 1.)

3 Some terminology

3.1 Trigonometric functions

Jyā, kōṭ-jyā (or *kōjyā*), and *utkrama-jyā* (or *śara*) (see Figure 2) are three trigonometric functions introduced by Indian astronomers and mathematicians. The earliest known Indian treatise containing references to these functions is Sūrya Sidhānta. These are functions of arcs of circles and not functions of angles. *jyā* and *kōti-jyā* are closely related to the modern trigonometric functions of sine and cosine. In fact, the origins of the modern terms of sine and cosine has been traced back to the Sanskrit words *jyā* and *kōti-jyā*.

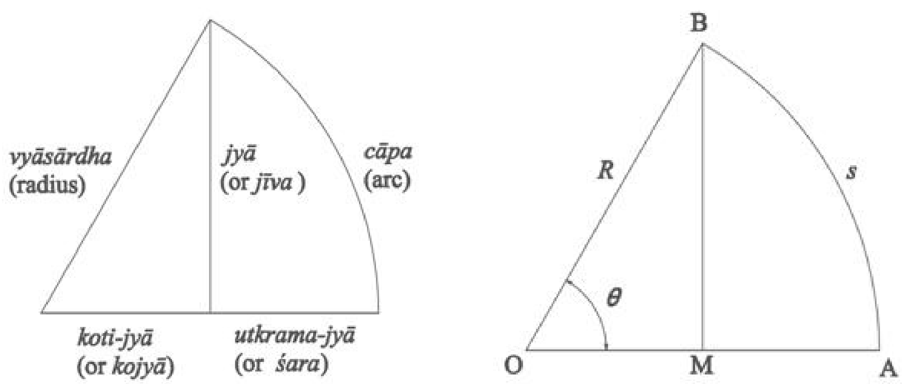

Figure 2: Diagram showing *cāpa*, *jyā*, etc.

Let 'arc AB' denote an arc whose two extremities are A and B of a circle with centre O. If a perpendicular BM be dropped from B to OA, then (see Figure 2):

- *jyā* of arc AB = BM
- *kōṭi-jyā* of arc AB = OM
- *utkrama-jyā* (or *śara* of arc AB) = MA

Let the radius of the circle be R and the length of arc AB be s, the angle subtended by arc AB at O measured in radians be $\theta = \frac{s}{R}$. Then we have (see Figure 2):

- *jyā* (arc AB)= $R\sin(\frac{s}{R})$
- *kō-jyā* (arc AB) $= R\cos(\frac{s}{R})$
- *utkrama-jyā* (arc AB) $= R(1 - \cos(\frac{s}{R}))$

3.2 *Kaṭapayādi* scheme

The *kaṭapayādi* scheme is a method for representing numbers using letters of the Sanskrit alphabet (see [9]). The inventor of this system of numeration is not known. But it had been used by Kerala astronomers and mathematicians as early as fifth century CE. The scheme was very popular among all sections of society in Kerala especially among astronomers and mathematicians. The system was virtually unknown in North India. The *kaṭapayādi* scheme is employed in Karantic musical traditions to code the serial number of a *melakarta rāga*.

There are several variations of the scheme. In one such scheme popular among members of the Kerala School of Astronomy and Mathematics, the assignment of digits to letters is as in Table 1.

1	2	3	4	5	6	7	8	9	0
ka	kha	ga	gha	ṅa	ca	cha	ja	jha	ña
ṭa	ṭha	ḍa	ḍha	ṇa	ta	tha	da	dha	na
pa	pha	ba	bha	ma	-	-	-	-	-
ya	ra	la	va	śa	ṣa	sa	ha	-	-

Table 1: Mapping of digits to letters in *kaṭapayādi* scheme

The following procedure was followed in representing large numbers in the *kaṭapayādi* scheme.

- Consonants have numerals assigned as per Table 1. For example, *ba* is always 3 whereas 5 can be represented by either *ṇa*, or *ma* or *śa*.
- All stand-alone vowels like *a* and *i* are assigned to 0.
- In case of a conjunct, consonants attached to a non-vowel will be valueless. For example, *kya* is formed by $k + y + a$. The only consonant standing with a vowel is *ya*. So the corresponding numeral for *kya* will be 1.
- Numbers are traditionally written in increasing place values from left to right. The number 386 which denotes $3 \times 100 + 8 \times 10 + 6$ in modern notations would be written as 683 in pre-modern Indian traditions.

Thus *Sadratnamāla* gives the value of π to 18 decimal places in the *kaṭapayādi* scheme as follows (see [9]):

bha-drā-ṃbu-ddhi-si-ddha-ja-nma-ga-ṇi-ta- śra-ddhā-sma-ya-d bhū-pa-gī
4 2 3 9 7 9 8 5 3 5 6 2 9 5 1 4 1 3

This denotes 423979853562951413 in Indian traditions and 314159265358979324 in modern notations.

4 Mādhava's value for π

To complete the numerical computations of the values of the sine and cosine functions, Mādhava needed accurate value of the mathematical constant π. The value of π mentioned in *Sadratnamāla* which is correct to 18 decimal places and which had been referred to above was not available to Mādhava. Mādhava had derived the following series for the computation of π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

This series is known in European mathematical traditions as the Gregory series and its discovery has variously been attributed to Gottfried Wilhelm Leibniz (1646 - 1716) and James Gregory (1638 1675). The series was known in Kerala more than two centuries before its European discovers were born (see [10]). Using this series and several correction terms Māadhava computed the following value for π (see p.119 [11]):

$$\pi = 3.1415926535922$$

5 Mādhava series for sine and cosine functions

None of Māadhava's works containing any of the series expressions attributed to him has survived. These series expressions are found in the writings of the followers of Mādhava in the Kerala school. At many places these authors have clearly stated that these are as told by Mādhava. Thus the enunciations of the various series found in Tantrasamgraha and its commentaries can be safely assumed to be in "Mādhava's own words". The translations of the relevant verses as given in the Yuktidīpika commentary of Tantrasamgraha (also known as Tantrasamgraha-vyākhya) by Śankara Vāriar (c.1500 - c.1560 CE) are reproduced below. These are then rendered in current mathematical notations (see [11]).

6 Mādhava's sine series

6.1 In Mādhava's own words

Mādhava's sine series is stated in verses 2.440 and 2.441 in Yukti-dīpika commentary (Tantra-samgra-ha-vyāakhya) by Śankara Vāriar. A translation of the verses follows (see p.114 [11]).

> Multiply the arc by the square of the arc, and take the result of repeating that (any number of times). Divide (each of the above numerators) by the squares of the successive even numbers increased by that number and multiplied by the square of the radius. Place the arc and the successive results so obtained one below the other, and subtract each from the one above. These together give the jiva, as collected together in the verse beginning with "*vidvān*" etc.

6.2 Rendering in modern notations

Let r denote the radius of the circle and s the arc-length.

- The following numerators are formed first:

$$s \cdot s^2, \quad s \cdot s^2 \cdot s^2, \quad s \cdot s^2 \cdot s^2 \cdot s^2, \ldots$$

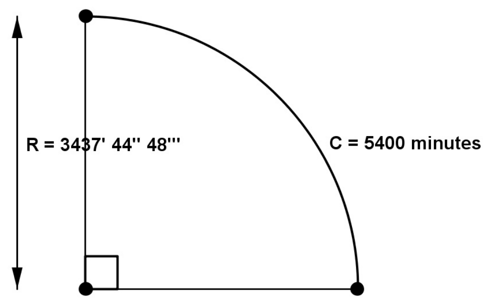

Figure 3: Mādhava's value of R

- These are then divided by quantities specified in the verse.

$$s\cdot\frac{s^2}{(2^2+2)r^2},\quad s\cdot\frac{s^2}{(2^2+2)r^2}\cdot\frac{s^2}{(4^2+4)r^2},\quad s\cdot\frac{s^2}{(2^2+2)r^2}\cdot\frac{s^2}{(4^2+4)r^2}\cdot\frac{s^2}{(6^2+6)r^2},\cdots$$

- Place the arc and the successive results so obtained one below the other, and subtract each from the one above to get the following expression for *jīva*:

$$\begin{aligned}\text{jīva} = s - \Big[s \cdot \frac{s^2}{(2^2+2)r^2} \\ - \Big[s \cdot \frac{s^2}{(2^2+2)r^2}\cdot\frac{s^2}{(4^2+4)r^2} \\ - \Big[s \cdot \frac{s^2}{(2^2+2)r^2}\cdot\frac{s^2}{(4^2+4)r^2}\cdot\frac{s^2}{(6^2+6)r^2} \\ - \cdots \quad \Big]\Big]\Big]\end{aligned}$$

6.3 Transformation to current notation

Let θ be the angle subtended by the arc s at the center of the circle. Then $s = r\theta$ and *jīva* $= r\sin\theta$. Substituting these in the last expression and simplifying we get

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

which is the infinite power series expansion of the sine function.

6.4 Mādhava's reformulation for numerical computation

The last line in the verse "as collected together in the verse beginning with *vidvān* etc." is a reference to a reformulation of the series introduced by Māadhava himself to make it convenient for easy computations for specified values of the arc and the radius (see p.116 - 117 [11]). For such a reformulation, Mādhava begins by considering an arc of a circle consisting of one quarter of a full circle. The measure of a quarter of a circle is taken as 5400 minutes (say C minutes). He then develops a scheme for the easy computations of the *jīva*-s of the various arcs of such a circle. Let R be the radius of a circle one quarter of which measures C (see Figure 3). Mādhava had already computed the value of π using his series formula for π. Using this value of π, namely 3.1415926535922, the radius R is computed as follows:

$$
\begin{aligned}
R =& 2 \times 5400/\pi \\
=& 3437.74677078493925 \\
=& 3437 \text{ arcminutes} \\
& 44 \text{ arcseconds} \\
& 48 \text{ sixtieths of an arcsecond} \\
=& 3437'44''48'''
\end{aligned}
$$

The expression for *jīva* can now be put in the following form:

$$
\begin{aligned}
\text{jīva} = s - \Big[& s \cdot \frac{s^2}{(2^2+2)R^2} \\
- \Big[& s \cdot \frac{s^2}{(2^2+2)R^2} \cdot \frac{s^2}{(4^2+4)R^2} \\
- \Big[& s \cdot \frac{s^2}{(2^2+2)R^2} \cdot \frac{s^2}{(4^2+4)R^2} \cdot \frac{s^2}{(6^2+6)R^2} \\
- & \quad \cdots \Big] \Big] \Big]
\end{aligned}
$$

Now, since $C = R \times \frac{\pi}{2}$, we have:

$$
\begin{aligned}
\frac{s^3}{R^2} &= \left(\frac{s}{R}\right)^3 R \\
&= \left(s \cdot \frac{\pi/2}{C}\right)^3 R \\
&= \left(\frac{s}{C}\right)^3 \cdot R \left(\frac{\pi}{2}\right)^3 \\
\frac{s^5}{R^4} &= \left(\frac{s}{R}\right)^5 R \\
&= \left(s \cdot \frac{\pi/2}{C}\right)^5 R \\
&= \left(\frac{s}{C}\right)^5 \cdot R \left(\frac{\pi}{2}\right)^5 \\
& \cdots
\end{aligned}
$$

No.	Expression	Value	In *kaṭapayādi* system
1	$R \times (\pi/2)^3/3!$	$2220'\ 39''\ 40'''$	*ni-rvi-ddhā-nga-na-rē-ndra-rung*
2	$R \times (\pi/2)^5/5!$	$273'\ 57''\ 47'''$	*sa-rvā-rtha-śī-la-sthi-ro*
3	$R \times (\pi/2)^7/7!$	$16'\ 05''\ 41'''$	*ka-vī-śa-ni-ca-ya*
4	$R \times (\pi/2)^9/9!$	$33''\ 06'''$	*tu-nna-ba-la*
5	$R \times (\pi/2)^{11}/11!$	$44'''$	*vi-dvāan*

Table 2: Mādhava's pre-computed coefficients in the power series for the sine function

Substituting these in the expression for *jīva* we have:

$$\text{jīva} = s - \left(\frac{s}{C}\right)^3 \left[\frac{R\left(\frac{\pi}{2}\right)^3}{3!} - \left(\frac{s}{C}\right)^2 \left[\frac{R\left(\frac{\pi}{2}\right)^5}{5!} - \left(\frac{s}{C}\right)^2 \left[\frac{R\left(\frac{\pi}{2}\right)^7}{7!} - \cdots \quad \right]\right]\right]$$

Mādhava now pre-computes the values given in Table 2. With these values, the computational scheme for the evaluation of the sine function takes the following form:

$$\begin{aligned}
\text{jiva} = s - \left(\frac{s}{C}\right)^3 &\Big[(2220'\ 39''\ 40''') \\
- \left(\frac{s}{C}\right)^2 &\Big[(273'\ 57''\ 47''') \\
- \left(\frac{s}{C}\right)^2 &\Big[(16'\ 05''\ 41''') \\
- \left(\frac{s}{C}\right)^2 &\Big[(33''\ 06''') \\
- \left(\frac{s}{C}\right)^2 &\Big[(44''') \\
- \cdots &\quad \Big]\Big]\Big]\Big]
\end{aligned}$$

7 Mādhava's cosine series

7.1 In Mādhava's own words

Māadhava's cosine series is stated in verses 2.442 and 2.443 in *Yukti-dīpika* commentary (also known as *Tantrasamgrahavyākhya*) by Śankara Vāriar. A translation of the verses follows (see p.115 [11]):

> Multiply the square of the arc by the unit (i.e. the radius) and take the result of repeating that (any number of times). Divide (each of the above numerators)

by the square of the successive even numbers decreased by that number and multiplied by the square of the radius. But the first term is (now) (the one which is) divided by twice the radius. Place the successive results so obtained one below the other and subtract each from the one above. These together give the *śara* as collected together in the verse beginning with *stena*, *stri*, etc.

7.2 Rendering in modern notations

Let r denote the radius of the circle and s the arc-length.

- The following numerators are formed first:

$$r \cdot s^2, \quad r \cdot s^2 \cdot s^2, \quad r \cdot s^2 \cdot s^2 \cdot s^2, \ldots$$

- These are then divided by quantities specified in the verse to obtain the following quantities:

$$\begin{aligned}
&r \cdot \frac{s^2}{(2^2-2)r^2},\\
&r \cdot \frac{s^2}{(2^2-2)r^2} \cdot \frac{s^2}{(4^2-4)r^2},\\
&r \cdot \frac{s^2}{(2^2-2)r^2} \cdot \frac{s^2}{(4^2-4)r^2} \cdot \frac{s^2}{(6^2-6)r^2},\\
&\ldots
\end{aligned}$$

- Place the arc and the successive results so obtained one below the other, and subtract each from the one above to get the following series for *śsara*:

$$\begin{aligned}
\text{śara} = & r \cdot \frac{s^2}{(2^2-2)r^2},\\
& -\Big[r \cdot \frac{s^2}{(2^2-2)r^2} \cdot \frac{s^2}{(4^2-4)r^2},\\
& -\Big[r \cdot \frac{s^2}{(2^2-2)r^2} \cdot \frac{s^2}{(4^2-4)r^2} \cdot \frac{s^2}{(6^2-6)r^2},\\
& -\cdots \quad \Big]\Big]
\end{aligned}$$

7.3 Transformation to current notation

Let θ be the angle subtended by the arc s at the centre of the circle. Then $s = r\theta$ and *śara* $= r(1 - cos\theta)$. Substituting these in the last expression and simplifying we get

$$1 - \cos\theta = \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots$$

which gives the power series expansion of the cosine function.

7.4 Mādhava's reformulation for numerical computation

The last line in the verse "as collected together in the verse beginning with *stena*, *stri*, etc." is a reference to a reformulation introduced by Mādhava himself to make the series convenient for easy computations for specified values of the arc and the radius (see p.118 [11]). As in the case of the sine series, M=aadhava considers a circle one quarter of which measures 5400 minutes (say C minutes) and develops a scheme for the easy computations of the *śara*-s of the various arcs of such a circle. Let R be the radius of a circle one quarter of which measures C. Then, as in the case of the sine series, Mādhava gets $R = 3437' \ 44'' \ 48'''$.

Māadhava's expression for *śara* corresponding to any arc s of a circle of radius R is equivalent to the following:

$$\begin{aligned}\text{śara} =& R \cdot \frac{s^2}{(2^2-2)R^2}, \\ & - \Big[R \cdot \frac{s^4}{(2^2-2)(4^2-4)R^4} \\ & - \Big[R \cdot \frac{s^6}{(2^2-2)(4^2-4)(6^2-6)R^6} \\ & - \cdots \quad \Big] \Big]\end{aligned}$$

Since $C = R \times (\pi/2)$, the above expression can be put in the following form:

$$\text{śara} = \left(\frac{s}{C}\right)^2 \left[\frac{R(\frac{\pi}{2})^2}{2!} - \left(\frac{s}{C}\right)^2 \left[\frac{R(\frac{\pi}{2})^4}{4!} - \left(\frac{s}{C}\right)^2 \left[\frac{R(\frac{\pi}{2})^6}{6!} - \cdots \quad \right] \right] \right]$$

Next Mādhava pre-computes the values given in Table 3. With these values, the computational scheme for the evaluation of the cosine function takes the following form:

$$\begin{aligned}\text{śara} = & \left(\frac{s}{C}\right)^2 \Big[(4241' \ 09'' \ 00''') \\ & - \left(\frac{s}{C}\right)^2 \Big[(872' \ 03'' \ 05''') \\ & - \left(\frac{s}{C}\right)^2 \Big[(071' \ 43'' \ 24''') \\ & - \left(\frac{s}{C}\right)^2 \Big[(03' \ 09'' \ 37''') \\ & - \left(\frac{s}{C}\right)^2 \Big[(05'' \ 12''') \\ & - \left(\frac{s}{C}\right)^2 \Big[(06''') \\ & - \cdots \quad \Big] \Big] \Big] \Big] \Big]\end{aligned}$$

No.	Expression	Value	In *kaṭapayādi* system
1	$R \times (\pi/2)^2/2!$	$4241'\ 09''\ 00'''$	*u-na-dha-na-kṛt-bhu-re-va*
2	$R \times (\pi/2)^4/4!$	$872'\ 03''\ 05'''$	*mī-nā-ngo-na-ra-siṃ-ha*
3	$R \times (\pi/2)^6/6!$	$071'\ 43''\ 24'''$	*bha-drā-nga-bha-vyā-sa-na*
4	$R \times (\pi/2)^8/8!$	$03'\ 09''\ 37'''$	*su-ga-ndhi-na-ga-nud*
5	$R \times (\pi/2)^{10}/10!$	$05''\ 12'''$	*strī-pi-śsu-na*
6	$R \times (\pi/2)^{12}/12!$	$06'''$	*ste-na*

Table 3: Mādhava's pre-computed coefficients in the power series for the cosine function

8 Mādhava's sine table

The table of values of the sine function prepared by Mādhava is given in Table 4 (see p.121 - 122 [11]). The entries in the fourth column of the Table are calculated using the entries in the third column of he Table. Let the eight digits in the third column of a row corresponding to an angle A be $d_1, d_2, \ldots, d_8$ read from left to right. According the conventions of the *kaṭapayādi* scheme, the corresponding value in the fourth column is computed as below:

$$\frac{\pi}{180 \times 60}\left[(1000d_8 + 100d_7 + 10d_6 + d_5) + \frac{10d_4 + d_3}{60} + \frac{10d_2 + d_1}{3600}\right]$$

9 Analysis of the algorithms

9.1 These are indeed algorithms!

In the traditional accounts of the history of Kerala mathematics emphasis is placed on the discovery of the power series expansions of the sine and cosine functions. The fact that the Kerala School had produced acceptable and sound proofs of these series expansions is also strongly emphasised. The methods and techniques used in the proofs have generated much interest and more controversy among the historians of mathematics, some arguing that these are the beginnings of calculus and Mādhava and his Kerala School should be accepted as the inventors of calculus while many others propounding that though the proofs are basically correct, the methods do not conclusively establish the invention of calculus. But, one fact that emerges from a close reading of the relevant passages is that the Kerala inventors of these series were more interested in the procedures that could be developed based on these series. For them these series provided nothing but an algorithm for the computation of the sine and cosine functions. This algorithmic aspects are all the more clear in the way the series expansions were formulated. It is in fact given as a step by step procedure for

Angle A in degrees	Mādhava's numbers for specifying $\sin A$ in ISO 15919 transliteration scheme	Mādhava's numbers for specifying $\sin A$ in Arabic numerals	Value of $\sin A$ derived from Mādhava's table	Modern value of $\sin A$
03.75	śreṣṭhō nāma variṣṭhānāṃ	22 05 4220	0.06540314	0.06540313
07.50	himādrirvēdabhāvanaḥ	85 24 8440	0.13052623	0.13052619
11.25	tapanō bhānu sūktajñō	61 04 0760	0.19509032	0.19509032
15.00	maddhyamaṃ viddhi dōhanaṃ	51 54 9880	0.25881900	0.25881905
18.75	dhigājyō nāśanaṃ kaṣṭaṃ	93 10 5011	0.32143947	0.32143947
22.50	channabhōgāśayāṃbikā	70 43 5131	0.38268340	0.38268343
26.25	mṛgāhārō narēśōyaṃ	53 82 0251	0.44228865	0.44228869
30.00	vīrō raṇajayōtsukaḥ	42 25 8171	0.49999998	0.50000000
33.75	mūlaḥ viśuddhaḥ nāḷasya	53 45 9091	0.55557022	0.55557023
37.50	gāneṣu viraḷā narāḥ	30 64 2902	0.60876139	0.60876143
41.25	aśuddhiguptā cōraśrīḥ	05 93 6622	0.65934580	0.65934582
45.00	śaṃkukarṇō nagēśvaraḥ	51 15 0342	0.70710681	0.70710678
48.75	tanūjō garbhajō mitraṃ	60 83 4852	0.75183985	0.75183981
52.50	śrīmānatra sukhī sakhē	25 02 7272	0.79335331	0.79335334
56.25	śaśī rātrou himāhārou	55 22 8582	0.83146960	0.83146961
60.00	vēgajñaḥ pathi sindhuraḥ	43 01 7792	0.86602543	0.86602540
63.25	chāya layō gajō nīlō	71 31 3803	0.89687275	0.89687274
67.50	nirmalō nāsti salkulē	05 30 6713	0.92387954	0.92387953
71.25	ratrou darpaṇamabhrāṃgaṃ	22 81 5523	0.94693016	0.94693013
75.00	nāgastuṃga nakhō balī	03 63 0233	0.96592581	0.96592583
78.75	dhīrō yuvā kathālōlaḥ	92 14 1733	0.98078527	0.98078528
82.50	pūjyō nārījanairbhagāḥ	11 02 8043	0.99144487	0.99144486
86.25	kanyāgārē nāgavallī	11 32 0343	0.99785895	0.99785892
90.00	devō viśvasthalī bhṛguḥ	84 44 7343	0.99999997	1.00000000

Table 4: Madhava's sine table

easy computations the function values. Moreover in Jyēṣṭhadēva's Yuktibhāṣa, the author has used precisely the term *kriya-krama* which clearly and unambiguously translates into procedure or, in modern terminology, an algorithm. Mādhava and other members of the Kerala School were fundamentally computationalists. Their goal was to compute the values of sine and cosine functions. For this they developed the necessary mathematics and then created the algorithm. The algorithm was then actually applied to create a table of values of the sine function correct to about eight decimal places. The procedures outlined above are Mādhava's algorithms for the computations of sine and cosine functions.

9.2 Use of a polynomial approximation

Mādhava's algorithms for the computation of sine and cosine functions use polynomial approximations. The algorithm for the computation of the sine function makes use of an 11 th degree polynomial where as the algorithm for the cosine function makes use of a 12 th degree polynomial. The orders of the polynomials were decided by the requirements of accuracy. They were targeting an accuracy to a third (that is, one-sixtieth of a second) which would suffice for their astronomical applications. The values computed by Mādhava could also be obtained by other specialized methods. But Mādhava did seek and get a general method speaks volumes about his computational genius.

9.3 Pre-computation of the coefficients

The observation that Mādhava was a computationaist par excellence would be reinforced if we note that he did not stop with inventing the power series expansions. He also developed an efficient computational scheme. One of the ingredients of this scheme was the pre-computation of the coefficients appearing in the polynomial approximations. The coefficients of both sine and cosine series were pre-computed. These precomputed coefficients are given in Table 2 and Table 3. Of course these are not in the modern decimal notations. They are in the methods and traditions then prevalent in Kerala.

9.4 Use of Horner's scheme

A final observation is that Mādhava had applied what is now known as the Horner's scheme for the computation of polynomials. The scheme is now attributed to William George Horner (1786 1837) who was a British mathematician and schoolmaster (see p.271 [12]). In modern notations, the scheme can be formulated as follows.

Given the polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n,$$

where a_0, a_1, ..., a_n are real numbers, let it be required to evaluate the polynomial at a specific value of x. To do this Horner suggested that the polynomial be expressed in the form:

$$p(x) = a_0 + x(a - 1 + x(a_2 + \cdots + x(a_{n-1} + a_n x) \cdots))$$

Horner's scheme requires only n multiplications and n additions to compute the value of a n-th degree polynomial. A quick glance at the expressions for sine and cosine using the pre-computed coefficients readily shows that Mādhava had actually implemented Horner's scheme in his algorithms.

Even though the algorithm is named after William George Horner, who described it in 1819, the method was already known to Isaac Newton in 1669, the Chinese mathematician Qin Jiushao in the 13th century, and even earlier to the Persian Muslim mathematician Sharaf al-Dn al-Ts in the 12 th century. The use of the scheme has also been traced to a Chinese work of the 3 rd century. However Mādhava's was the first conscious and deliberate application of the scheme in a computational algorithm with the intention of reducing the complexity of numerical procedures.

9.5 Simultaneous computation of sine and cosine

In many modern implementations of routines for the calculations of the sine and cosine functions, there would be one routine for the simultaneous computation of sine and cosine. The argument in favour of such an approach is that in most engineering and other applications, whenever sine or cosine is required, the other would also be required. So a common algorithm which returns both values simultaneously would be more time efficient and economical especially if the main component of a certain computational process is the evaluation of sine and cosine functions. It would appear that Mādhava had anticipated such a scenario. This is evidenced by the description of one common procedure for the evaluation of sine and cosine functions in Yuktibhās. a. We quote a translation of this common procedure below (see p.102 [13]):

> This being the case, the following is the procedure (kriya-krama) to be adopted. The required is the first result. When this squared, halved and divided by the radius, the second result is got. Keep this second result separately. Now multiply the second result also by the arc and divide by 3 and also by radius. Place the result got below the first result. Then multiply this also by the ar and divide by four and the radius. Keep the result below the second result. in this manner, derive successive results by multiplying the previous result by the arc and dividing by corresponding successive numbers 1,2,3, etc. and by radius. Now place below the first result the odd results, viz. the third, the fifth, etc. and place below the second result the even results, viz, the fourth, the sixth, etc. Then subtract successively from the bottom result from the one above it, the remainder from the one still above it. Ultimately in the first column the resultant first result will be left and in the second column the resultant second result will be left. These will be the required Rsine (jyā) and Rversine (śara).

10 Comparison with modern algorithms

A comparison of Mādhava's scheme with a modern scheme for the computation of sine and cosine values is worth considering. One such algorithm openly available in the internet is the programme sincos.c included in many Linux distributions. For discussion we have taken the programme included in the Open64 Compiler developed by Computer Architecture and Parallel Systems Laboratory in University of Delaware (see [14]). One can also compare Mādhava's scheme with other codes for the computation of these functions, for example, the codes given in the Netlib Repository maintained by AT&T Bell Laboratories, the University of Tennessee and Oak Ridge National Laboratory (see [15], [16]). These programs are not using the polynomials used by Mādhava. Instead they are using the minimax polynomial computed using the Remez algorithm to improve the accuracy of computations. But the fact remains that these modern algorithms are indeed using polynomial approximations for computations of the sine and cosine functions. Also, they are using pre-computed coefficients and Horner's scheme for the evaluation of polynomials. Since the polynomials used are the minimax polynomials, the pre-computed coefficients given in these computer programs are different from the pre-computed coefficients employed by Mādhava.

10.1 Coefficients for evaluation of sine

The following program segment specifies the pre-computed coefficients in the polynomial approximation for the sine function. The values are given in the IEEE:754 floating point format.

```

/* coefficients for polynomial
approximation of sin on +/- pi/4 */

static const du S[] =

D(0x3ff00000, 0x00000000),
D(0xbfc55555, 0x55555548),
D(0x3f811111, 0x1110f7d0),
D(0xbf2a01a0, 0x19bfdf03),
D(0x3ec71de3, 0x567d4896),
D(0xbe5ae5e5, 0xa9291691),
D(0x3de5d8fd, 0x1fcf0ec1),
;

```

10.2 Coefficiets for the evaluation of cosine

The following program segment specifies the pre-computed coefficients in the polynomial approximation for the cosine function. The values are also given in the IEEE:754 floating point format.

```
/* coefficients for polynomial approximation of cos on +/- pi/4 */

static const du C[] =

D(0x3ff00000, 0x00000000),
D(0xbfdfffff, 0xffffff96),
D(0x3fa55555, 0x5554f0ab),
D(0xbf56c16c, 0x1640aaca),
D(0x3efa019f, 0x81cb6a1d),
D(0xbe927df4, 0x609cb202),
D(0x3e21b8b9, 0x947ab5c8),
;
```

10.3 Polynomial approximations using Horner's scheme

The following lines in the code describes the computations of the polynomial approximations for the sine and cosine functions simultaneously.

```
xsq = x*x;

cospoly = (((((C[6].d*xsq + C[5].d)*xsq +
C[4].d)*xsq + C[3].d)*xsq +
C[2].d)*xsq + C[1].d)*xsq + C[0].d;

sinpoly = (((((S[6].d*xsq + S[5].d)*xsq +
S[4].d)*xsq + S[3].d)*xsq +
S[2].d)*xsq + S[1].d)*(xsq*x) + x;
```

10.4 Comparison

It is true that this programme having more than 500 lines of code has made use of several other ideas as well. But the critical components continues to be the following which are essentially the ideas enshrined in Mādhava's computational scheme developed more than six centuries ago.

- Use of an approximating polynomial.
- Pre-computation of coefficients.

- Use of Horner's scheme for the evaluation of polynomials.
- Simultaneous computation of sine and cosine functions.

11 Conclusion

Sangamagrāma Mādhava, the founder of Kerala School of Astronomy, was a great computationalist and an algorithmist. When confronted with a numerical problem he created the necessary mathematical tools with rigorous proofs, developed efficient computational schemes, and implemented those schemes. Mathematicians and astronomers have studied the contributions of the Kerala School in great detail. But it appears that his works have not been thoroughly scrutinised by algorithmists and computationalists. There is a need of a more intensive study of the methods of Kerala School from an algorithmic point of view. This will help us understand our own past much better. More importantly it may help us develop new algorithmic paradigms which may lead to the creation of newer computational tools.

References

[1] G. G. Joseph, *A Passage to Infinity: Medieval Indian Mathematics from Kerala and Its Impact*. New Delhi: Sage Publications Pvt. Ltd, 2009.

[2] C. M. Whish, "On the hindu quadrature of the circle and the infinite series of the proportion of the circumference to the diameter exhibited in the four sastras, the tantra sahgraham, yucti bhasha, carana padhati and sadratnamala," Transactions of the Royal Asiatic Society of Great Britain and Ireland (Royal Asiatic Society of Great Britain and Ireland, vol. 3 (3), pp. 509 523, 1834.

[3] I. S. B. Murthy, *A modern introduction to ancient Indian mathematics*. New Delhi: New Age International Publishers, 1992.

[4] V. J. Katz,*The mathematics of Egypt, Mesopotemia, China, India and Islam: A source book*. Princeton: Princeton University Press, 2007, ch. Chapter 4 : Mathematics in India IV. Kerala School (pp. 480 - 495).

[5] K. Plofker, *Mathematics in India*. Princeton, NJ: Princeton University Press, 2010.

[6] Madhava of sangamagramma. [Online]. Available: http://wwwgap. dcs.st-and.ac.uk/ history/Projects/Pearce/Chapters/Ch9 3.html

[7] K. V. Sarma and V. S. Narasimhan, "Tantrasamgraha," Indian Journal of History of Science, vol. 33 (1), Mar. 1998.

[8] K. V. Sarma and S. Hariharan, "Yuktibhasa of jyesthadeva : a book of rationales in indian mathematics and astronomy - an analytical appraisal," Indian Journal of History of Science, vol. 26 (2), pp. 185 207, 1991.

[9] A. V. Raman, "The katapayadi formula and the modern hashing technique," Annals of the History of Computing, vol. 19 (4), pp. 4952, 1997.

[10] R. Roy, "The discovery of the series formula for π by Leibniz, Gregory, and Nilakantha in Sherlock Holmes in Babylon and other tales of mathematical history", R. W. Marlow Anderson, Victor Katz, Ed. The Mathematical Association of America, 2004.

[11] C. K. Raju, *Cultural foundations of mathematics : The nature of mathematical proof and the transmission of the calculus from India to Europe in the 16th c. CE*. Delhi: Centre for Studies in Civilizations, 2007.

[12] F. Cajori, *A history of mathematics*, 5th ed. Chelsea Publishing Series, 1999.

[13] Jyēṣṭhadēva, Gaṇita-yukti-bhs̄. a, K. V. Sarma, Ed. New Delhi: Hindustan Book Agency, 2008.

[14] `osprey/libm/mips/sincos.c. [Online]. Available: http:// www.open64.net/ doc/d6/d08/ sincos 8c-source.html`

[15] `kin.c. [Online]. Available: http://www.netlib.org/ fdlibm/ k sin.c`

[16] `k cos.c. [Online]. Available: http://www.netlib.org/ fdlibm/ k cos.c`

Appendix

11.1 Transliteration of Indic scripts

Figure 4 shows the scheme for transliterating the letters in Devanagari script according to the International Alphabet of Sanskrit Transliteration (IAST). This is based on a standard established by the International Congress of Orientalists at Geneva in 1894. For the most part, IAST is a subset of ISO 15919. However there are some minor differences between IAST and ISO 15919.

The script of Malayalam language, the language of *Yuktibhaṣha*, has letters which are not present in the Devanagari script. Figure 5 shows these letters and their transliterations as per ISO 15919 scheme.

TechS Vidya e-Journal of Research
ISSN 2322 - 0791
Vol. 1 (2012-13) pp.88-108.

अ	आ	इ	ई	उ	ऊ	ऋ	ऌ
a	ā	i	ī	u	ū	ṛ	ḷ
ए	ऐ	ओ	औ	अं	अः		
e	ai	o	au	ṃ	ḥ		

क	ख	ग	घ	ङ
ka	kha	ga	gha	ṅa
च	छ	ज	झ	ञ
ca	cha	ja	jha	ña
ट	ठ	ड	ढ	ण
ṭa	ṭha	ḍ	ḍha	ṇa
त	थ	द	ध	न
ta	tha	da	dha	na
प	फ	ब	भ	म
pa	pha	ba	bha	ma

य	र	ल	व
ya	ra	la	va
श	ष	स	ह
ś	ṣ	sa	ha

Figure 4: International Alphabet of Sanskrit Transliteration

ഏ	ഓ	ള	റ	ഴ
ē	ō	ḷa	ṟa	ḻa

Figure 5: Transliteration of Malayalam letters not present in Devanagari script