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A brief review of the equations of solid mechanics[1]

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Salutations

Let me offer with humility salutations[2] to all my teachers, most of all to the One Real Innermost Guru, the Guru of all gurus, the real source of everything beautiful and sublime.

1 Introduction

This article is meant to be a brief review of the governing equations of Solid Mechanics addressed primarily to B.Tech. graduates in Aerospace/ Civil/ Mechanical Engineering who have already had a course in Advanced Mechanics of Solids and/ or the Theory of Elasticity. The purpose of such a review is (i) to serve as a recapitulation of the nature of a revisit (Governing Equations of Solid Mechanics Revisited), and (ii) thus, to set the stage the stage for the finite element formulation of stress analysis/ structural engineering problems.

1.1 Why should we learn theory when computers are available?

Some people seem to hold the opinion that it is no more necessary to learn the theory, now that computers and software packages are readily available. Nothing is farther from the truth. Computers just cannot be trusted without the knowledge, understanding, appreciation, experience and judgement of the engineer computist. The data handled and the speed

[1] *Editors' comments: This is a preliminary version of an extensive discussion of the equations of mechanics being prepared by the author of the article. The Editors of TechS Vidya e-Journal of Research acknowledge their thankfulness to the author for kindly permitting them to publish the article in the present form. The copyright of the paper still rests with the author and it is being published only for strictly academic purposes and for limited (internal) circulation.*

[2] It is an Indian tradition to remember with pleasure and gratitude one's teachers and to offer one's salutations. Let me too follow this great tradition.

of operation can be so mind boggling, voluminous and fast that one is often tempted to give these computers a larger than life image. These factors underscore the absolute necessity to model the physical problems and to formualate them in a way suitable for machine computation.

Some others feel that it is now pointless to learn the several closed form solutions so laboriously and ingeniously worked out in the nineteenth century and in the first half of the twentieth century. One might justifiably ask: if most, if not all, really important problems can be solved only by numerical procedures, why do we have to learn the several closed form solutions? Why should we bother to learn the so-called *exact* solutions which are often based on idealised situations? Such solutions are exact only in the sense that the governing mathematical equations are satisfied exactly. These *exact* solutions are almost always of *approximate* problems. Often an approximate solution of an *exact* problem (or, at least, of a more realistically formulated problem) is no worse than (often far superior to) an *exact* solution.

1.2 A partial answer

The answer to the question raised is in the form of stating these reasons: (i) it improves our sharpness of intellect and analytical abilities; (ii) the study of these *exact* solutions of idealised problems leaves a residue in the form of sophistication and finesse in our thinking; and (iii) we would become aware of the many complications that we would otherwise have overlooked, or failed to notice[3].

It is perhaps true that the various techniques of solutions painstakingly developed have partially lost their relevance. This is both a relief and a pity. Some of the techniques are notoriously laborious (What a relief!), while some are aesthetically beautiful (What a pity!).

When all these factors are taken into reckoning, the governing equations still hold the centre stage position. They continue to be the heart and soul of the problem formulation. It is essential to understand them with conceptual clarity.

We shall now take a close look at the governing equations, sometimes referred to as the field equations of the classical theory of elasticity, or the statics of deformable, elastic bodies erroneously called strength of materials for a long, long time.

2 The governing equations

The governing equations can be classified under three heads:

(a) the (differential) equations of equilibrium;

(b) the strain-displacement relations; and

(c) the constitutive equations.

[3] We do not demand, or even suggest, that one should suspect a rabbit in every bush. Yet only one who is exposed to different situations can detect complications lurking in the background.

The compatibility equations are not part of the governing equations in this context. See pages 8, 20 and 21 for some more information and explanatory remarks.

We shall discuss each set of these equations one by one.

3 Differential equations of equilibrium

The stress components σ_{ij} (9 components, reducing to 6 components because of the symmetry of the stress matrix, $\sigma_{ij} = \sigma_{ji}$) are required to satisfy the (differential) equations of equilibrium (6 equations, reducing to 3 if the moment equations, which are responsible for the symmetry of the stress matrix, are disregarded).

$$\sigma_{ij,j} + F_i = 0 \qquad i, j = 1, 2, 3. \tag{1}$$

(We have used the index notation in this equation: summation over the repeated index j. Here i and j take the values $i, j = 1, 2, 3.$. Also, F_i denotes the body force per unit volume and the comma (,) indicates differentiation.)

Being a tensor equation, this is valid in all coordinate systems. In general, the comma $(,)$ is to be understood as standing for covariant differentiation. In the rectangular Cartesian system, this will reduce to the usual partial derivative. The Christoffel symbols, we recall, are all zero in such a highly simplified coordinate system.

3.1 In a rectangular Cartesian system

Thus, in a rectangular Cartesian system the above equation (really three equations) reads as

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0; \tag{2a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0; \tag{2b}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0. \tag{2c}$$

We note that with the symmetry of the stress matrix $\sigma_{ij} = \sigma_{ji}$ taken into account, there are six (6) unknowns σ_{ij} and 3 equations. It is clear from this observation that there are more unknowns than there are equations. Thus, *every* problem in stress analysis is statically indeterminate internally. We know that the equations of equilibrium alone cannot determine the stress components inside a body; we need to have additional equations.

3.2 In other coordinate systems

The differential equations of equilibrium in other coordinate systems such as, for example, the cylindrical polar coordinate system can be obtained (a) by drawing an elemental block, marking all the stress components on the various faces, computing the corresponding forces by multiplying each of these stress components by the relevant area, and considering the net force along each of the three directions and setting it equal to zero; (b) by transforming the equations in the rectangular Cartesian coordinates (x, y, z) to, say, the cylindrical polar coordinates (r, θ, z) by using the mathematical transformations; or (c) by writing the equation in tensor form and working out the appropriate form in, say, the cylindrical polar coordinates (which involves working out the metric tensor g_{ij}, computing the Christoffel symbols and writing out the tensor equation $\sigma_{ij,j} + F_i = 0$ in long hand interpreting the comman $(,)$ as the symbol for covariant differentiation). When the indicated processes are completed, we obtain the desired equations.

3.3 In a cylindrical polar coordinate system

The differential equations in, say, the cylindrical coordinates (r, θ, z) will appear as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0; \tag{3a}$$

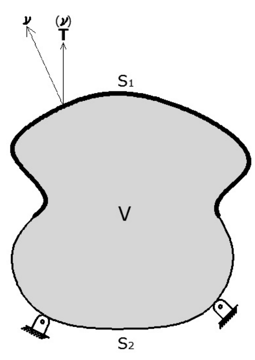
$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0; \tag{3b}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + F_z = 0. \tag{3c}$$

One way to derive the differential equations of equilibrium is outlined below.

3.4 Derivation of differential equations of equilibrium

A body of volume V enclosed by a surface of area S acted upon by some surface tractions $\overset{(\nu)}{T_i}$ and body forces F_i is shown in Fig. 1. The applied forces and moments and the support conditions are such that the body is in static equilibrium. The equations of equilibrium,



The body is adequately supported, and is in static equilibrium. In other words, the body forces and the surface tractions are statically compatible. $\overset{(\nu)}{\mathbf{T}}$ will not, in general, be along the normal ν. It is not possible to specify both the tractions and the corresponding work absorbing displacements at any point on the boundary.

Figure 1: A body in static equilibrium with surface tractions and body forces

therefore, are

$$\iint_S \overset{(\nu)}{T_i}\, dS + \iiint_V F_i\, dV = 0, \quad (i = 1, 2, 3). \tag{4}$$

Using Cauchy's result $\overset{(\nu)}{T_i} = \sigma_{ji} n_j \ \ (= \sigma_{ij} n_j)$ and changing the surface integral to a volume integral, we obtain

$$\iint_S \overset{(\nu)}{T_i}\, dS + \iiint_V F_i\, dV = \iint_S \sigma_{ji} n_j\, dS + \iiint_V F_i\, dV = \iiint_V (\sigma_{ij,j} + F_i)\, dV \tag{5}$$

which should vanish for every arbirary volume V, small or large. This requirement that the volume integral should vanish for every volume V leads to the conclusion that the integrand itself should vanish at every point inside V. Thus, we obtain $\sigma_{ij,j} + F_i = 0$ at every point inside V.

We shall now turn to the next set of governing equations, viz., the strain-displacement relations.

4 Strain-displacement relations

These are also known as the kinematic relationships. The strain components e_{ij} (9 reduced to 6 because of symmetry, $e_{ij} = e_{ji}$) are related to the displacement components u, v, w by the following six equations.

$$e_{xx} = \frac{\partial u}{\partial x}; \qquad e_{yy} = \frac{\partial v}{\partial y} : \qquad e_{zz} = \frac{\partial w}{\partial z}; \tag{6a}$$

$$e_{xy} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]; \quad e_{yz} = \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right]; \quad e_{zx} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]. \tag{6b}$$

4.1 Compatibility: integrability

The equations (6a) relate the normal strains e_{xx}, e_{yy} and e_{zz} to the (partial derivates of) the displacement components u, v and w, while the ones (6b) relate half[4] the shearing strains to the (partial derivatives[5] of) the displacement components.

The six strain components have common parentage; they have all come from the three displacement components (u, v, w). Thus, the six e_{ij}'s are not independent; they are

[4] The learned professor, Dr Bhoj Raj Seth (B.R. Seth), for long at IIT, Kharagpur, used to emphasise the absolute necessity of introducing the factor of half in these equations. We should realise that without this factor of half, the strain matrix will not have the transformation properties enjoyed (and required) by the strain tensor. A comparison of the transformation equations of stress components and strain components would give us this insight.

Let me pause here to pay homage to this great teacher of ours with much pleasure and gratitude. What a great inspiration even his mere presence was!

[5] 'crossed' derivatives: u with y, v with x, and so on

cousins or half brothers. The e_{ij}'s are, therefore, related to one another. These relationships among the six strain components are the compatibility equations. Mathematically speaking, the above six strain-displacement relations may be regarded as a system of six partial differential equations for the determination of only three unknown functions (displacement components) u, v and w. Thus, this is an overdetermined system, and consequently this system cannot have any solution in general; certain conditions must be satisfied so that there can be admissible (single-valued, continuous) functions $u.v, w$. These integrability conditions are the compatibility equations (or conditions).

4.2 Rotations

Associated with the strain components at a point are the rotations ω_{ij}. These are the components of half of the curl of the displacement vector. The strain components e_{ij} are symmetric ($e_{ij} = e_{ji}$), while the rotation components are skew-symmetric ($\omega_{ij} = -\omega_{ji}$). Thus, $\omega_{xx} = \omega_{yy} = \omega_{zz} = 0$.

$$\omega_{xx} = 0; \qquad \omega_{yy} = 0 : \qquad \omega_{zz} = 0; \tag{7a}$$

$$\omega_{xy} = \frac{1}{2}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right]; \quad \omega_{yz} = \frac{1}{2}\left[\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right]; \quad \omega_{zx} = \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]. \tag{7b}$$

These rotations are not related to the stress components in Solid Mechanics and, therefore, they have only a secondary role here. In Fluid Mechanics, on the other hand, they play a crucial role.

4.3 Kinematic relations in index notations

The above equations can be written compactly in index notation as

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}); \qquad \omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}); \quad [u_{i,j} = e_{i,j} + \omega_{i,j}]. \tag{8}$$

As remarked earlier, these commas (,) refer to the usual partial differentiation when rectangular Cartesian coordinates are used. In the general curvilinear coordinnate system, these commas stand for covariant differentiation. Thus, to obtain the strain-displacement relations in, say, cylindrical polar coordinates, the metric tensor g_{ij}, and then the Christoffel symbols have to be worked out, and the equations written out in long hand with the commas intrepreted as covariant differentiation. When the indicated steps or operations are carried out in full, the strain-displacement relations in cylindrical polar coordinates (r, θ, z) appear as

$$e_{rr} = \frac{\partial u_r}{\partial r}; \qquad e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} : \qquad e_{zz} = \frac{\partial u_z}{\partial z}; \tag{9a}$$

$$e_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right]; \;\; e_{\theta z} = \frac{1}{2}\left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right]; \;\; e_{zr} = \frac{1}{2}\left[\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right]. \tag{9b}$$

Here u_r, u_θ, u_z are the displacement components u_i in the cylindrical polar coordinates (r, θ, z).

In some books, say, for example, in Timoshenko & Goodier, Theory of Elasticity, u, v, w are taken as the displacement commponents in the (r, θ, z) directions, respectively. In this case, the strain displacement components appear as

$$e_{rr} = \frac{\partial u}{\partial r}; \tag{10a}$$

$$e_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{10b}$$

$$e_{zz} = \frac{\partial w}{\partial z}; \tag{10c}$$

$$\gamma_{r\theta} = 2e_{r\theta} = \gamma_{\theta r} = 2e_{\theta r} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}; \tag{10d}$$

$$\gamma_{\theta z} = 2e_{\theta z} = \gamma_{z\theta} = 2e_{z\theta} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}; \tag{10e}$$

$$\gamma_{zr} = 2e_{zr} = \gamma_{rz} = 2e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{10f}$$

4.4 Two-dimensional axisymmetric case

The simplified case of two-dimensional asisymmetric problems is of special interest. Now in the polar coordinates (r, θ), the above strain-displacement equations get simplified as

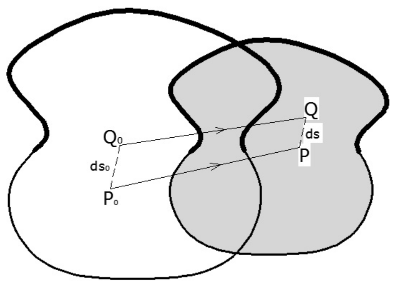
$$e_{rr} = \frac{du_r}{dr}; \qquad e_{\theta\theta} = \frac{u_r}{r}. \tag{11}$$

We should not fail to notice that, for this axisymmetric case, the displacements are now only in the radial direction; there are no tangential displaements. Even though the tangential *displacements* u_θ are everywhere zero, the tangential *strain* $e_{\theta\theta} = u_r/r$ is not zero! Den Hartog in his characteristic inimitable humour remarks[6]: " the first one is fairly obvious, and the second one refers to the feelings of a middle-aged gentleman who lets out one notch of his belt after his good dinner."

4.5 Large displacements: finite elasticity theory

When the displacements are *large* as in finite elasticity theory, the original and the current configurations of a body will be quite different. Now we have a choice: the indepenedent variables may be taken either as (x, y, z) referred to the original (undeformed) configuration, or as (ξ, η, ζ) referred to the current (deformed) configuration. Accordingly, we may write in the following two ways, and obtain the strain-displacement relations in two different ways.

[6] J.P. Den Hartog: Advanced Strength of Materials, McGraw-Hill, 1952.



Unshaded: original configuration
Shaded: deformed (current) configuration

In problems of finite elasticity, the initial and the final (current) configurations can be quite different.

Figure 2: Original and deformed configurations

The material particle $P_0(x, y, z)$ moves, on deformation of the body, to the place specified by $P(\xi, \eta, \zeta)$. The displacements are u, v, w. These may be written as functions of (x, y, z) or of (ξ, η, ζ). A neighbouring material particle $Q_0(x + dx, y + dy, z + dz)$ moves to the place $Q(\xi + d\xi, \eta + d\eta, \zeta + d\zeta)$. Accordingly, P_0Q_0 of original (undeformed) length ds_0 changes to PQ of final (current) length ds. A nearby material particle $Q_0(x + dx, y + dy, z + dz)$ moves to the place $Q(\xi + d\xi, \eta + d\eta, \zeta + d\zeta)$.

The difference in length between ds_0 and ds is a measure of the deformation. Writing the expression $\frac{1}{2}\left[(ds)^2 - (ds_0)^2\right]$, which is a measure of the deformation, in terms of (x, y, z) or in terms of (ξ, η, ζ), we obtain the following equations. The first is in terms of the variables (particle labels) (x, y, z), while the second is in terms of the position, or place, variables (ξ, η, ζ).

$$\begin{aligned}\frac{1}{2}\left[(ds)^2 - (ds_0)^2\right] &= e_{xx}(dx)^2 + e_{yy}(dy)^2 + e_{zz}(dz)^2 \\ &\quad + e_{xy}(dx)(dy) + e_{yz}((dy)(dz) + e_{zx}(dz)(dx) \qquad (12) \\ &= E_{\xi\xi}(d\xi)^2 + E_{\eta\eta}d\eta^2 + E_{\zeta\zeta}(d\zeta)^2 \\ &\quad + E_{\xi\eta}(d\xi)(d\eta) + E_{\eta\zeta}(d\eta)(d\zeta) + E_{\zeta\xi}(d\zeta)(d\xi) \qquad (13)\end{aligned}$$

The strain-displacement relations are the following:

(a) Lagrangian (independent variables x, y, z)

$$e_{xx} = u_x + \frac{1}{2}\left(u_x^2 + v_x^2 + w_x^2\right) \qquad (14a)$$

$$e_{yy} = v_y + \frac{1}{2}\left(u_y^2 + v_y^2 + w_y^2\right) \qquad (14b)$$

$$e_{zz} = w_z + \frac{1}{2}\left(u_z^2 + v_z^2 + w_z^2\right) \qquad (14c)$$

$$e_{xy} = \frac{1}{2}\left[(u_y + v_x) + (u_x u_y + v_x v_y + w_x w_y)\right] = e_{yx} \qquad (14d)$$

$$e_{yz} = \frac{1}{2}\left[(v_z + w_y) + (u_y u_z + v_y v_z + w_y w_z)\right] = e_{zy} \qquad (14e)$$

$$e_{zx} = \frac{1}{2}\left[(w_x + u_z) + (u_z u_x + v_z v_x + w_z w_x)\right] = e_{xz} \tag{14f}$$

The subscript denotes partial differentiation, with respect to the subscript concerned, as, for example: $u_x \equiv \dfrac{\partial u}{\partial x}$; and $u_x u_y \equiv \dfrac{\partial u}{\partial x} \times \dfrac{\partial u}{\partial y}$.

(b) Eulerian (independent variables ξ, η, ζ)

$$E_{xx} = u_\xi - \frac{1}{2}\left(u_\xi^2 + v_\xi^2 + w_\xi^2\right) \tag{15a}$$

$$E_{yy} = v_\eta - \frac{1}{2}\left(u_\eta^2 + v_\eta^2 + w_\eta^2\right) \tag{15b}$$

$$E_{zz} = w_\zeta - \frac{1}{2}\left(u_\zeta^2 + v_\zeta^2 + w_\zeta^2\right) \tag{15c}$$

$$E_{\xi\eta} = \frac{1}{2}\left[(u_\eta + v_\xi) - (u_\xi u_\eta + v_\xi v_\eta + w_\xi w_\eta)\right] = E_{\eta\xi} \tag{15d}$$

$$E_{\eta\zeta} = \frac{1}{2}\left[(v_\zeta + w_\eta) - (u_\eta u_\zeta + v_\eta v_\zeta + w_\eta w_\zeta)\right] = E_{\zeta\eta} \tag{15e}$$

$$E_{\zeta\xi} = \frac{1}{2}\left[(w_\xi + u_\zeta) - (u_\zeta u_\xi + v_\zeta v_\xi + w_\zeta w_\xi)\right] = E_{\xi\zeta} \tag{15f}$$

The subscript again denotes partial differentiation, with respect to the subscript concerned, as, for example: $u_\xi \equiv \dfrac{\partial u}{\partial \xi}$; and $u_\xi u_\eta \equiv \dfrac{\partial u}{\partial \xi} \times \dfrac{\partial u}{\partial \eta}$.

[**Caution:** $\dfrac{\partial u}{\partial x}$ is computed when u is regarded and written as a function of x, y, z. On the other hand, $\dfrac{\partial u}{\partial \xi}$ is computed when u is regarded and written as a function of ξ, η, ζ. These two functions $u(x, y, z)$ and $u(\xi, \eta, \zeta)$ have two different functional forms. Strictly speaking, if we write $u = u(x, y, z)$, we must write

$$u = u(x, y, z) = u(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) = u^*(\xi, \eta, \zeta).]$$

Let us note that in the infinitesimal, linear theory, the nonlinear terms are all disregarded. Now, in this simplified case, there is no distinction between the Lagrangian and Eulerian measures of strain. Then, in this case,

$$u_{xx} = u_{\xi\xi} = \frac{\partial u}{\partial x}; \qquad v_{xx} = v_{\eta\eta} = \frac{\partial v}{\partial y}; \qquad w_{zz} = w_{\zeta\zeta} = \frac{\partial w}{\partial z}. \tag{16}$$

In index notation, the strain-displacement equations read

$$e_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i} + u_{l,i}u_{l,j}] \qquad \text{(Lagrangian);} \tag{17}$$

$$E_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i} - u_{l,i}u_{l,j}] \qquad \text{(Eulerian).} \tag{18}$$

As indicated above, it is not a good idea to use the same letter u when the indepenedent variables are (x, y, z), and also when they are (ξ, η, ζ).

When the deformations are large, it is almost indispensable to employ general tensors, and not just Cartesian tensors. Thus, if u_i and U_i are the displacement vectors from P_0 to P referred to the frames of reference A and B,

$$e_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i} + u_{l,i}u^l{}_{,j}] \qquad \text{(Lagrangian);} \tag{19}$$

$$E_{ij} = \frac{1}{2}[U_{i,j} + U_{j,i} - U_{l,i}U^l{}_{,j}] \qquad \text{(Eulerian).} \tag{20}$$

Let us note that these are different from those for the classical, linear theory by adding (+) or subtracting (-) a *'small'* nonlinear term.

Having discussed the strain-displacement relations, we shall next consider the constitutive equations.

5 Constitutive equations

These are also known as the material law, Hooke's law, or the generalised Hooke's law in the restricted case of the classical linear theory of elasticity.

The nine (9) stress components (9 reduced to 6 because of symmetry) are related to the nine (9) strain components (9 reduced to 6 because of symmetry again). In the simplest case of infinitesimal, linear, isotropic, elastic materials (which alone we consider for the most part here), these are given by the following equations.

5.1 Generalised Hooke's law: strains in terms of stresses

$$e_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right], \tag{21a}$$

$$e_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{zz} + \sigma_{xx}\right)\right], \tag{21b}$$

$$e_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right], \tag{21c}$$

$$\gamma_{xy} \equiv 2e_{xy} = 2e_{yx} \equiv \gamma_{yx} = \frac{\tau_{xy}}{G}, \tag{21d}$$

$$\gamma_{yz} \equiv 2e_{yz} = 2e_{zy} \equiv \gamma_{zy} = \frac{\tau_{yz}}{G}, \tag{21e}$$

$$\gamma_{zx} \equiv 2e_{zx} = 2e_{xz} \equiv \gamma_{xz} = \frac{\tau_{zx}}{G}, \tag{21f}$$

where E and G are the modulii of elasticity and rigidity, respectively.

5.2 Stresses in terms of strains

The above equations may be inverted to give the stress components in terms of the strain components as

$$\sigma_{xx} = \frac{\nu E}{(1+\nu)(1-2\nu)}\,e + \frac{E}{1+\nu}\,e_{xx}, \tag{22a}$$

$$\sigma_{yy} = \frac{\nu E}{(1+\nu)(1-2\nu)}\, e + \frac{E}{1+\nu}\, e_{yy}, \tag{22b}$$

$$\sigma_{zz} = \frac{\nu E}{(1+\nu)(1-2\nu)}\, e + \frac{E}{1+\nu}\, e_{zz}, \tag{22c}$$

$$\tau_{xy} = G\gamma_{xy}, \tag{22d}$$

$$\tau_{yz} = G\gamma_{yz}, \tag{22e}$$

$$\tau_{zx} = G\gamma_{zx}, \tag{22f}$$

where $e \equiv e_{xx} + e_{yy} + e_{zz}$ is the volumetric strain (the first invariant of the strain tensor e_{ii}).

5.3 In terms of Lamé's constants

It is sometimes conveneient to write these in the form

$$\sigma_{xx} = \lambda e + 2Ge_{xx}, \tag{23a}$$

$$\sigma_{yy} = \lambda e + 2Ge_{yy}, \tag{23b}$$

$$\sigma_{zz} = \lambda e + 2Ge_{zz}, \tag{23c}$$

$$\tau_{xy} = G\gamma_{xy} = 2Ge_{xy}, \tag{23d}$$

$$\tau_{yz} = G\gamma_{yz} = 2Ge_{yz}, \tag{23e}$$

$$\tau_{zx} = G\gamma_{zx} = 2Ge_{zx}, \tag{23f}$$

in terms of the Lamé's constants λ and G, which are related to the Young's modulus of elasticity, E and the Poisson's ratio, ν by the equations

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \tag{24a}$$

$$G = \frac{E}{2(1+\nu)}. \tag{24b}$$

We note that there are only two (2) independent elastic constants for a linear, elastic, isotropic material. These are usually taken as E and ν by engineers. Applied mathematicians and elasticians sometimes prefer to work in terms of the Lamé's constants λ and G. The constitutive equations, that is, the generalised Hooke's law in this case, may be written in index notation as

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2Ge_{ij}, \tag{25}$$

where δ_{ij} is the Kronecker delta.

5.4 Anisotropy and orthotropy

Materials cannot always be treated as isotropic. Some materials have intrinsically different properties along different directions. Cold rolled copper and wood are two cases in point.

The properties are different along the directions of rolling. Similarly, the properties along the grains are distinctly different from those perpendicular to this direction. Sometimes the method of construction introduces anisotropy in a structure. If ribs are provided in one direction, but not in the other, a slab or a plate, made up of essentially an isotropic material, will exhibit anisotropy. More significantly, composites, which have in recent times become an important structural material, are decidedly anisotropic. To deal with such materials, we need to consider anisotropic elasticity also.

If the nine (9) stress components are related to the nine (9) strain components by the linear equations $e_{ij} = c_{ijkl}\sigma_{kl}$, we can see that there are $9 \times 9 = 81$ elastic constants e_{ijkl} $(i, j, k, l = 1, 2, 3;\quad 3^4 = 81)$. If the symmetry conditions $\sigma_{ij} = \sigma_{ji}$ and $e_{ij} = e_{ji}$ are invoked, this number 81 reduces to 36 $(6 \times 6 = 36)$. If, furthermore, the existence of a strain energy density function is assumed, this number 36 reduces to 21. Thus, in the general case of anisotropy, there are 21 elastic constants.

To continue with such simplifications or reductions, if there is one plane of elastic symmetry, this number will be reduced further to 13. If the material has three planes of elastic symmetry, there will be only nine (9) elastic constants. These are called orthotropic materials. (Many types of bio-membranes such as cell walls should be modeled as orthotropic.) And finally, when the material is isotropic, this number nine (9) reduces further to three (3). Even these three (3) are not independent. Thus, we may conclude that there are just two (2) independent elastic constants for a linear, elastic, isotropic material. These, as indicated earlier, are usually taken as E and ν, or the Lamé's constants λ and G.

We shall next recapitulate an important principle widely used in mechanics, viz., the principle of virtual work as applied to a deformable body.

6 Principle of virtual work

We shall now consider the principle of virtual work in a form that is applicable to a deformable body. This equation that will be discussed in this section perhaps does not enjoy the same status as a governing equation as the ones in the three sets of equations discussed above. Nevertheless, it is a general principle that has far reaching implications, and is the basis of several energy principles of mechanics. In this sense, it is an equation of fundamental importance.

Let us recall from elementary mechanics the principle of virtual work applied to rigid bodies, and to deformable bodies made up of rigid elements, such as mechanisms. This principle, when applied to rigid bodies, does not give us anything new or particularly useful[7], but it is of great use when applied to mechanisms. This principle is equally valid for deformable bodies discussed in the mechanics of solids. However, it is to be recast in a form that can be applied to solid mechanics. We shall undertake such an exercise below.

[7] J.P. Den Hartog: Mechanics, dover Publications, New York, 1948. We advise the readers to be sure to read what this extraordinarily humorous author has to state in this context (about the Duke of Malborough who marched his army of twenty thousand men up a hill, and marched them down again!)

Cosider a deformable body in static equilibrium acted upon by a body force distribution F_i acting throughout its volume V, and surface traction $\overset{(\nu)}{T_i}$ acting throughout the surface $S = S_1 + S_2$. See Fig. 1. On one part S_1 of the boundary, the surface tractions $\overset{(\nu)}{T_i}$ are prescribed and, therefore, known. On the other remaining part S_2, the displacements u_i are prescribed and, therefore, known. (We may point out that it is not possible, in general, to prescribe both the surface traction and the displacements in advance on any part of the boundary; that is, here the surface traction on S_2, nor the displacements on S_1.)

If a virtual displacement field δu_i is applied, the various forces undergoing the virtual displacements will perform some virtual work. It is emphasised that during the small hypothetical[8] virtual displacements, (i) the applied forces are all held constant, and that (ii) the virtual displacements must be consistent with the prescribed (i.e., specified or given) displacement boundary conditions and the constraints. This second condition demands that $\delta u_i = 0$ on S_2. The principle of virtual work states that the total virtual work done by all the forces, the external ones (body forces and tractions) which are externally applied on the body), and the internal ones (stresses which are internally developed inside the body) during these virtual displacements is zero.

It is more convenient, however, to compute the *external* virtual work, and to show that it is equal[9] to the *internal* virtual work. This is what we propose to do below.

Accordingly, a virtual displacement (actually a virtual displacement field) δu_i is imposed on the body, and the *external* virtual work is computed as follows.

$$\delta W_{virtual} = \iiint\limits_V F_i\, \delta u_i\, dV + \iint\limits_S \overset{(\nu)}{T_i}\, \delta u_i\, dS \quad \text{(index notation; summation over } i). \tag{26}$$

The second integral may cover either S_1 only, or the entire surface S. It makes no difference, because δu_i has to be zero on S_2. No variation is permissible on S_2, as the displacements u_i are prescribed there.

When Cauchy's result, viz., $\overset{(\nu)}{T_i} = \sigma_{ji} n_j \;\; (= \sigma_{ij} n_j)$ is used in the second integral, the above equation (26) reads

$$\delta W_{virtual} = \iiint\limits_V F_i\, \delta u_i\, dV + \iint\limits_S \sigma_{ij} n_j\, \delta u_i\, dS \tag{27}$$

$$= \iiint\limits_V F_i\, \delta u_i\, dV + \iiint\limits_V (\sigma_{ij}\, \delta u_i)_{,j}\, dV \tag{28}$$

$$= \iiint\limits_V F_i\, \delta u_i\, dV + \iiint\limits_V [\sigma_{ij}\, (\delta u_i)_{,j} + \sigma_{ij,j}\, (\delta u_i)]\, dV \tag{29}$$

[8]This must be seen correctly in the light of the Calculus of Variations in the background.

[9]It may appear contradictory to state that (i) the sum of the external and internal virtual work is zero, and again that (ii) the external virtual work done is equal to the internal virtual work done. There is no conflict or contradiction here; the correct algebraic signs are also to be taken into our reckoning.

$$= \iiint\limits_V (F_i + \sigma_{ij,j})\, \delta u_i \, dV + \iiint\limits_V \sigma_{ij}\, (\delta u_i)_{,j} \, dV. \tag{30}$$

The surface integral in Eq.(27) is replaced by the corresponding volume integral in Eq.(28) using Gauss' theorem. Also, the second term in the second integral in Eq.(29) is transferred to the first integral in Eq.(30).

The first integral vanishes, because $\sigma_{ij,j} + F_i = 0$ (equations of equilibrium). We need to process the second integral to obtain the result in the desired convenient form. Towards this end, let us introduce a kinematically compatible strain field variation δe_{ij}.

We note that the operators δ and $(,)$ commute; that is, the order of the operations of variation δ and of partial differentiation $(,)$ is interchangeable. Thus,

$$(\delta u_i)_{,j} = \delta(u_{i,j}) = \delta(e_{ij} + \omega_{ij}) = \delta e_{ij} + \delta \omega_{ij} \tag{31}$$

where we have written $u_{i,j} = e_{ij} + \omega_{ij}$. See Eq.(8).

Consider now the integrand in the second integral of Eq.(30):

$$\sigma_{ij}\, (\delta u_i) = \sigma_{ij}\, [\delta e_{ij} + \delta \omega_{ij}] = \sigma_{ij}\, \delta e_{ij}. \tag{32}$$

Let us recall that (i) the stress tensor σ_{ij} is symmetric $\sigma_{ij} = \sigma_{ji}$), (ii) the rotation tensor ω_{ij} is skew-symmetric $(\omega_{ij} = -\omega_{ji})$; and that, therefore, (iii) the product $\sigma_{ij}\, \delta\omega_{ij} = 0$.

$$\begin{aligned} \sigma_{ij}\, \delta\omega_{ij} &= \sigma_{ji}\, \delta\omega_{ij} && \text{(symmetry of } \sigma_{ij}) \\ &= -\sigma_{ji}\, \delta\omega_{ji} && \text{(skew-symmetry of } \omega_{ij}) \\ &= -\sigma_{ij}\, \delta\omega_{ij} && \text{(interchange of } i \text{ and } j;\ \text{both dummy indices).} \end{aligned} \tag{33}$$

The term $\sigma_{ij}\, \delta\omega_{ij}$ is equal to $-\sigma_{ij}\, \delta\omega_{ij}$, showing that each of them must be zero. Thus, we arrive at the result in the desired form as

$$\iiint\limits_V F_i\, \delta u_i \, dV + \iint\limits_S \overset{(\nu)}{T_i}\, \delta u_i \, dS = \iiint\limits_V \sigma_{ij}\, \delta e_{ij} \, dV. \tag{34}$$

One way to interpret this equation is to realise that the left hand side may be regarded as the external virtual work, and that the right hand side as the internal virtual work.

It is, of course, understood that the body force distributions F_i and the surface tractions $\overset{(\nu)}{T_i}$ are statically compatible in the sense that the body is in static equilibrium.

We note that a *necessary* condition for static equilibrium is that the external virtual work is equal to the internal virtual work done for *any* kinematically compatible (admissible) deformable field $(\delta u_i,\ \delta e_{ij})$, as long as the body forces F_i and the surface tractions $\overset{(\nu)}{T_i}$ are statically compatible. We can show that this condition is also *sufficient* for static equilibrium.

A very significant, and perhaps surprising, fact is that this equation holds, no matter what the constitutive equation is! The constitutive equations connecting the stress components σ_{ij} and the strain components e_{ij} do not enter into it.

7 Some general observations

We shall make a few general observations here which would clarify the overall picture.

7.1 Materials, models and constitutive equations

Not all materials can be regarded as elastic. There are many other models where the material will have to be considered as plastic, viscoelastic, thermoelastic, etc. In all these cases, the material law (that is, the constitutive equations) will be entirely different. More complicated constitutive equations will have to be employed when we need to consider materials with memory, etc.

It is repeated for emphasis: whether a material is elastic, plastic or viscoelastic is not an intrinsic property of the material. Concrete, for example, may be considered as an elastic material for some purposes. However, if long term creep effects have to be considered, the concrete, the very same concrete, will now have to be treated (or modeled) as a viscoelastic material. Stated differently, the words elastic, plastic, viscoelastic, etc., refer not so much to the *material*, but to the *model* that we employ. The model chosen should be in accordance with our objective.

In many cases, the constitutive relations are extremely difficult to find and write down. Blood and bio-fluids present major challenges in this regard. It is difficult even to suggest the form of the constitutive equations. It is even more difficult to obtain experimentally the numerical values of the constants or parameters that appear in the proposed constitutive equations.

Comprehensive theories on constitutive equations have been developed. There are some basic, fundamental ideas. The principle of material frame indifference, often referred to as the principle of objectivity, places some restrictions on the constitutive relationships. These considerations are quite abstract. These cannot be discussed here.

7.2 (a) and (b) same for all models

It is important to realise that the first two sets of equations, that is, (a) the differential equations of equilibrium, and (b) the strain-displacement relations, are the same for all materials (more appropriately referred to as for all models). It is the third set of equations (c), and the third set (constitutive equations) alone, that are different for other materials (models) such as plastic, viscoelastic and thermoelastic materials (models).

Even though the strain-displacement relations are the same for all materials (models), there could be differences in the actual equations used. In the classical, infinitesimal, linear theory of elasticity, it may be sufficient to retain only the linear terms, and the strain-displacement relations accepted as

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{35}$$

When large strains are encountered, the nonlinear terms also may have to be included.

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{l,i}u_{l,j}) \tag{36}$$

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{37}$$

and its companion equations. (See pp. 10-13, or better still, refer to a standard book on finite elasticity theory.)

Similarly, when large strains and strain rates have to be considered as in plasticity, other measures of strain are sometimes employed. In the early days of development of the theory of plasticity, it was common to use the so-called *true strain* (logarithmic strain). (There is nothing *true* about such a measure of strain, nor anything *false* about other measures of strain.)

7.3 General comments

A word or two may be added about the Poisson's ratio ν. A realistic value is about 0.3 for many engineering materials like steel. Some materials like beryllium and concrete have such low values of ν that they are sometimes considered as zero for simplifying calculations. Several equations become simpler if the Poisson's ratio ν is set equal to zero. The upper limit of ν is 0.5, because we may accept that, for example, a hydrostatic state of compression cannot possibly lead to an increase in volume! However, anomalous behaviour does occur now and then. Strange situations have come up in recent years. In plasticity theory, we often assume incompressibility; this is equivalent to assuming that the Poisson's ratio $\nu = 0.5$.

We have seen that, in general, there are 15 governing equations to determine 15 unknown functions. These unknowns, which are all functions of position, are (i) the stress components σ_{ij} (9 reduced to 6 because of symmetry); (ii) the strain components e_{ij} (9 reduced to 6 because of symmetry); and (iii) the displacement components u_i (3). The governing equations available to determine the aforesaid 15 unknown functions are (a) the differential equations of equilibrium (3); (b) the strain-displacement relations (6); and (c) the constitutive equations (here for the classical, linear theory of elasticity, the generalised Hooke's law) (6). The unknowns are $6+6+3 = 15$ in number, while the equations available to determine them are also $3 + 6 + 6 = 15$.

Thus, the general problem of having to determine 15 unknown functions from the available 15 equations, several of them differential equations, is far too difficult even for the best mathematicians. Even though some of the finest mathematicians had been labouring assiduously on these problems during the last 250 years or even more, the problem is still far from being solved completely. Several remarkable advances have, no doubt, been made. Yet these are not anywhere near a successful solution of the general problem. Thus, it is essential to look for numerical methods of solution. One of the most promising methods of solution, perhaps the most successful one, is the finite element method.

We shall make a few comments on the compatibility equations.

7.4 Compatibility equations and their role

The compatibility equations are not part of the governing equations in this formulation. Their role is merely to ensure that the displacements are admissible functions (single-valued, continuous[10]). If the displacement components are assumed, then the compatibility equations have no further role to play. This is the case in, say, St. Venant's semi-inverse method to solve the torsion problem (and, indeed, the more general bending problem). In the traditional finite element formulation[11], a displacement field inside a finite element (in terms of the nodal values) is assumed. Here too, therefore, the compatibility equations do not have to be considered. Another way of stating this fact is that the compatibility equations will now be automatically satisfied.

The compatibility equations in rectangular Cartesian coordinates are the following. They are six (6) in number, three in each of the two groups shown below.

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\,\frac{\partial^2 e_{xy}}{\partial x \partial y} \tag{38a}$$

$$\frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} = 2\,\frac{\partial^2 e_{yz}}{\partial y \partial z} \tag{38b}$$

$$\frac{\partial^2 e_{zz}}{\partial x^2} + \frac{\partial^2 e_{xx}}{\partial z^2} = 2\,\frac{\partial^2 e_{xz}}{\partial x \partial z} \tag{38c}$$

$$\frac{\partial^2 e_{zz}}{\partial x \partial y} + \frac{\partial^2 e_{xy}}{\partial z^2} = \frac{\partial^2 e_{yz}}{\partial z \partial x} + \frac{\partial^2 e_{zx}}{\partial y \partial z} \tag{38d}$$

$$\frac{\partial^2 e_{yy}}{\partial x \partial z} + \frac{\partial^2 e_{xz}}{\partial y^2} = \frac{\partial^2 e_{xy}}{\partial y \partial z} + \frac{\partial^2 e_{yz}}{\partial x \partial y} \tag{38e}$$

$$\frac{\partial^2 e_{xx}}{\partial y \partial z} + \frac{\partial^2 e_{yz}}{\partial x^2} = \frac{\partial^2 e_{xz}}{\partial x \partial y} + \frac{\partial^2 e_{xy}}{\partial x \partial z} \tag{38f}$$

In index notation, these appear in capsule form as

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0, \quad (i,j,k,l = 1,2,3). \tag{39}$$

Although there are $3 \times 3 \times 3 \times 3 = 81$ equations here, only six (6) of them, the ones shown above in long hand in rectangular Cartesian coordinates, are independent.

The compatibility equations will play the crucial role of a governing equation when the stresses (stress distributions) are proposed as possible solutions. A case in point is the solution of two-dimensional problems using the Airy's stress function. The Airy's stress function $\phi = \phi(x, y)$ is so cleverly defined that the equations of equilibrium are automatically satisfied. Now the question is this: will any choice of $\phi = \phi(x, y)$ solve the problem? The answer is a firm no.

[10]They may have to be differentiable too so that the strain components exist. We cannot examine such mathematical conditions rigorously here.

[11]In recent years, there have been mixed formulations. The compatibility conditions are important when the stresses are assumed without considering if the resulting displacements match.

7.5 Two essential requirements

Why is this so? We shall examine the situation carefully. There are primarily two requirements. (i) One, the stress distribution must satisfy the equations of equilibrium. This condition is automatically satisfied because of the clever definition of the Airy's stress function. (ii) Now the second requirement is the following. These stress distributions will correspond to strain distributions which, in turn, will correspond to displacement distributions. These displacement distributions shall be compatible. When the Airy's stress function $\phi = \phi(x, y)$ is assumed, the first condition is clearly satisfied, while nothing is known about the second condition, the condition of the compatibility of displacements. Thus, in such problems, the Airy's stress function $\phi = \phi(x, y)$ must satisfy the biharmaonic equation $\bigtriangledown^4\phi = 0$, which is really the statement of compatibility of the resulting displacements.

This situation is often the source of difficulty in elasticity problems. Two conditions shall be uncompromisingly satisfied: (i) the equations of equilibrium, and (iii) the compatibility of displacements. The first is on the stresses or stress distributions, while the second is on the displacements or displacement distributions.

We shall not pursue the matter further here except to state that there are several approaches that address this difficulty, the most important of which are the energy theorems.

7.6 Other aspects

In addition to what is stated so far, thermodynamics also plays a role in the mechanics of solids. This role is more than nominal in, say, problems in the theory of plasticity. Considerable energy is dissipated during plastic flow, and naturally enough, thermodynamics, often irreversible thermodyanmics, assumes importance.

We have not examined various issues, important though they are, here. Uniqueness of solution, St. Venant's principle, etc. are some of them. Even more important is the question: are the stresses inside a body independent of the material of the body? This question is not easy to answer comprehensively covering all situations. Under what conditions are the stresses independent of the material? This question and its answer are of prime importance in, say, photoelasticity. If the stresses are dependent on the material, it is pointless to determine experimentally the stresses in a model made up of a photoelastic material to know the stresses in, say, a dam made of concrete.

7.7 An interesting observation

We shall point out below an interesting feature which would throw additional light on some aspects related to our topic.

We had, we may recall, classified the governing equations (on p. 5) into (a), (b) and (c). They can be rewritten again in a slightly different form.

(a) The (differential) equations of equilibrium,

(b) the strain-displacement relations, and

(c) the constitutive equations.

These are, respectively,

$$L_1\boldsymbol{\sigma} = \boldsymbol{F}, \tag{40a}$$

$$L_2\boldsymbol{u} = \boldsymbol{e}, \tag{40b}$$

$$L_3\boldsymbol{e} = \boldsymbol{\sigma}. \tag{40c}$$

where L_1, L_2 and L_3 are three linear[12] operators.

In two dimensions, the equations (40a) and (40b) are,

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \tag{41a}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}. \tag{41b}$$

We can note the curious relationship[13] between the two linear operators L_1 and L_2, viz., $L_1 = L_2^T$!

The same interesting relationship can be seen in three dimensions also.

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau yz \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \tag{42a}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{bmatrix} \tag{42b}$$

[12] in the usual classical, linear theory of elasticity

[13] I acknowledge with much pleasure and gratitude the interesting personal discussions with Dr Gangan Prathap and private correspondence with Dr Somenath Mukherjee on this topic, even though I happen to have slight differences of opinion with these very learned scientists who are specialists in the area of finite element methods.

For an isotropic material with $\lambda = E\nu/[(1+\nu)(1-2\nu)]$ (where E, G and ν are, respectively, the Young's modulus of elasticity, the shear modulus and the Poisson's ratio), the constitutive equation (40c) can also be written in the following form.

$$\begin{bmatrix} (\lambda+2G) & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & (\lambda+2G) & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & (\lambda+2G) & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2e_{xy} \\ 2eyz \\ 2e_{zx} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau yz \\ \tau_{zx} \end{bmatrix} \tag{43}$$

The above equations give us a clue of when the stiffness matrix in a finite element formulation can be expected to be symmetric. If the operator L_3 in Eq. (40c) is symmetric, (that is. the 6×6 matrix in Eq. (43) is symmetric, $L_3^T = L_3$), we can obtain the following relation.

$$\begin{aligned} \boldsymbol{F} &= L_1 \boldsymbol{\sigma} \\ &= L_1 [L_3 \boldsymbol{e}] \\ &= L_1 [L_3 (L_2 \boldsymbol{u})] \\ &= [L_1 L_3 L_2] \boldsymbol{u} \\ &= \left[L_2^T L_3 L_2\right] \boldsymbol{u}, \end{aligned}$$

indicating that, as $L_1 = L_2^T$, the relationship

$$\boldsymbol{F} = \left[L_2^T L_3 L_2\right] \boldsymbol{u} \tag{44}$$

is symmetric whenever the operator L_3 is symmetric. This equation tells us that the stiffness matrix will be symmetric when the constitutive operator L_3 is symmetric (which is usually, but not necessarily always, the case).

8 Closure

Given above is a brief recapitulation of the governing equations of Solid Mechanics. We shall have to be satisfied with this in the context of a review of the governing equations as a prerequisite to learn the finite element method.

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